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Logarithmic corrections to black hole entropy and AdS/CFT correspondence

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Abstract: We calculate the correction to the Bekenstein-Hawking entropy formula for five dimensional AdS-Schwarzschild black holes due to thermodynamic fluctuations. The result is then compared with the boundary gauge theory entropy corrections via AdS/CFT correspondence. We then further generalise our analysis for the rotating black hole in five dimensional AdS space.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence.

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In recent years, new insights of space-time physics have emerged particularly through realising the idea of holography in terms of AdS/CFT correspondence [1]–[4]. According to this correspondence, type-IIB string theory in a five dimensional AdS space times a five-sphere is dual to the large N limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. More interestingly, in [5], it was further argued that the gauge theory in question, at high temperature, is dual to AdS-Schwarzschild black hole at the same temperature. This indeed leads to a new interpretation of Hawking-Page phase transition in terms of transition between the confining and de-confining phase of Yang-Mills gauge theory. It further follows from [5] that the thermodynamic variables of CFT can be read off from the boundary of AdS-Schwarzschild black hole. In particular, at high temperature, CFT calculation gives the correct entropy of the black hole up to a constant numerical multiplicative factor.

Recently, there are several works in the literature [6]–[10] suggesting that for a large class of black holes (AdS-Schwarzchild being one), the Bekenstein-Hawking area-entropy law receives additive logarithmic corrections due to thermal fluctuations. Typically, the corrected formula has the form

$$\mathcal{S} = S_0 - c \log S_0 \,, \tag{1}$$

where S_0 is the standard Bekenstein-Hawking term and c is a number. The purpose of this note is to analyse this correction of the entropy of AdS-Schwarzschild black hole in the light of AdS/CFT correspondence. In particular, in this note, we calculate the correction to the entropy of CFT due to thermal fluctuations and then compare it with the entropy correction of the black hole (1) following AdS/CFT dictionary.

The logarithmic correction to the entropy due to thermal fluctuation comes out to be of the form given in (1) with $c = 1/2$. We show that this matches with what is expected from AdS/CFT correspondence. However, this number differs from what was found in existing literature, see for example [11].[1]

In [11], the correction to the Bekenstein-Hawking entropy of AdS-Schwarzschild black hole has been calculated within canonical ensemble. We first briefly review the calculation in grand canonical setup.[2]

The partition function in the grand canonical ensemble is given by

$$Z(\alpha, \beta) = \int_0^\infty \int_0^\infty \rho(n, E) e^{(\alpha n - \beta E)} dn dE \,, \tag{2}$$

where $\alpha = \beta\mu$, μ is the chemical potential and $\beta = 1/T$ is the inverse temperature. Here we have set $k_B = 1$ so that the temperature is having the dimension of energy. The density of states, $\rho(n, E)$, can be obtained from the above equation by inverse Laplace transformation [12], as

$$\rho(n, E) = \left(\frac{1}{2\pi i}\right)^2 \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} Z(\alpha, \beta) e^{(-\alpha n + \beta E)} d\alpha d\beta \,,$$

[1] We thank Parthasarathi Majumdar for several fruitful e-mail correspondence with us in this regard.

[2] However, later in the calculation, we will set the chemical potential to zero, making this equivalent to canonical ensemble.

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$$= \left(\frac{1}{2\pi i}\right)^2 \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} e^{S(n,E,\alpha,\beta)} d\alpha d\beta\,, \tag{3}$$

where the function $S(n, E, \alpha, \beta)$ is defined through

$$S(n, E, \alpha, \beta) = \ln Z(\alpha, \beta) - \alpha n + \beta E\,. \tag{4}$$

The integral in (3) will be performed by the saddle-point approximation and the main contribution to this integral will come around the equilibrium point (α_0, β_0), where the integrand is stationary, and the stationary conditions are determined by $\frac{\partial S(n,E,\alpha,\beta)}{\partial \alpha}|_{(\alpha_0,\beta_0)} = 0$, and $\frac{\partial S(n,E,\alpha,\beta)}{\partial \beta}|_{(\alpha_0,\beta_0)} = 0$, which amounts to the following two equations

$$\begin{aligned} \frac{\partial \ln Z}{\partial \alpha}|_{(\alpha_0,\beta_0)} - n &= 0\,, \\ \frac{\partial \ln Z}{\partial \beta}|_{(\alpha_0,\beta_0)} + E &= 0\,. \end{aligned} \tag{5}$$

Expanding the exponent in the integrand of (3) to second order around the point (α_0, β_0), we get a gaussian integral. Finally, after performing the integral we obtain

$$\rho(n, E) = \frac{e^{S(\alpha_0,\beta_0)}}{2\pi\sqrt{M}}\,, \tag{6}$$

where M is the determinant of the following matrix:

$$M = \begin{pmatrix} \frac{\partial^2 \ln Z}{\partial \alpha^2}|_{(\alpha_0,\beta_0)} & \frac{\partial^2 \ln Z}{\partial \alpha \partial \beta}|_{(\alpha_0,\beta_0)} \\ \frac{\partial^2 \ln Z}{\partial \beta \partial \alpha}|_{(\alpha_0,\beta_0)} & \frac{\partial^2 \ln Z}{\partial \beta^2}|_{(\alpha_0,\beta_0)} \end{pmatrix}. \tag{7}$$

Evaluating the determinant (with the constraint that the values of the off diagonal elements of the matrix M are small in comparison to the diagonal elements) we get the following from of the density of states:

$$\rho(n, E) = \frac{e^{S(\alpha_0,\beta_0)}}{2\pi\sqrt{\frac{\partial^2 \ln Z}{\partial \alpha^2}|_{(\alpha_0,\beta_0)} \times \frac{\partial^2 \ln Z}{\partial \beta^2}|_{(\alpha_0,\beta_0)}}}\,. \tag{8}$$

Hence, the entropy is

$$\mathcal{S} \equiv \ln(\mathcal{E}\rho) = S(\alpha_0, \beta_0) + \ln \frac{\mathcal{E}}{\sqrt{\frac{\partial^2 \ln Z}{\partial \beta^2}|_{(\alpha_0,\beta_0)}}} + \text{higher order terms}\,. \tag{9}$$

It can be seen that ρ in our calculation has a dimension of inverse temperature. To get entropy as a log of dimensionless quantity, we have multiplied the density of state with $\mathcal{E}$ which has the dimension of energy. This scale $\mathcal{E}$ is set by the particular system in question. Since $\frac{\partial^2 \ln Z}{\partial \beta^2}|_{(\alpha_0,\beta_0)} = C_v T^2$, where C_v is the specific heat at constant volume, the correction to the entropy becomes:

$$\mathcal{S} = S(\beta_0) + \ln \frac{\mathcal{E}}{\sqrt{C_v T^2}} + \cdots \tag{10}$$

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In (9) and (10), we have set the chemical potential to zero. Furthemore, the scale $\mathcal{E}$ has been set to be the temperature T of the system. This is because we have temperature as the only available scale in canonical ensemble. Hence, the above equation becomes

$$\mathcal{S} = S(\beta_0) - \frac{1}{2}\ln C_v + \cdots \tag{11}$$

Having obtained the general form of correction to entropy, we would like to apply it for black holes. Note that, for (11) to make sense, C_v has to be positive. In the context of black hole, it means that the variation of mass with respect to temperature is positive. Five dimensional AdS-Schwarzschild black hole does indeed satisfy this property in certain choice of parameters. The metric of such black hole is given by

$$ds^2 = -\left(1 - \frac{16\pi M G_5}{3\Omega_3 r^2} + \frac{r^2}{l^2}\right) dt^2 + \left(1 - \frac{16\pi M G_5}{3\Omega_3 r^2} + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\,, \tag{12}$$

where $d\Omega_3^2$ is the metric on unit sphere S^3 and Ω_3 is the area of that unit sphere. G_5, M and $\Lambda = -6/l^2$ are five dimensional Newton's constant, mass and the cosmological constant. The horizon of the black hole is at

$$r^2 = r_+^2 = \frac{l^2}{2}\left(-1 + \sqrt{1 + \frac{48\pi M G_5}{3\Omega_3 l^2}}\right). \tag{13}$$

The entropy, temperature, and the specific heat are given by

$$\begin{aligned}
S_{\mathrm{BH}} &= \frac{\Omega_3 r_+^3}{4G_5}\,, \\
T &= \frac{4r_+^2 + 2l^2}{4\pi l^2 r_+} = \frac{1}{\pi}\left(\frac{\Omega_3}{4G_5}\right)^{\frac{1}{3}} \times \left(1 + \frac{2r_+^2}{l^2}\right) S_{\mathrm{BH}}^{-\frac{1}{3}}\,, \\
C &= \frac{dM}{dT_H} = \frac{\partial M}{\partial r_+}\frac{\partial r_+}{\partial T_H} = 3\frac{2r_+^2 + l^2}{2r_+^2 - l^2} S_{\mathrm{BH}}\,.
\end{aligned} \tag{14}$$

If

$$r_+^2 > \frac{l^2}{2}\,, \tag{15}$$

the specific heat becomes positive and evaluating the mass, temperature and specific heat of black hole, in this case, we get:

$$\begin{aligned}
M &\sim S_{\mathrm{BH}}^{\frac{4}{3}}\,, \\
T_H = \frac{1}{\pi}\left(\frac{\Omega_3}{4G_5}\right)^{\frac{1}{3}} \times \frac{2r_+^2}{l^2} S_{\mathrm{BH}}^{-\frac{1}{3}} &\sim S_{\mathrm{BH}}^{\frac{1}{3}}\,, \\
C &\sim 3S_{\mathrm{BH}}\,.
\end{aligned} \tag{16}$$

On substituting the above values of mass, temperature and specific heat into equation (11), we get the corrected entropy as

$$\mathcal{S}_{\mathrm{BH}} = S_{\mathrm{BH}} - \frac{1}{2}\ln S_{\mathrm{BH}} + \cdots \tag{17}$$

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According to AdS/CFT correspondence, on the boundary of AdS, we have conformally invariant $\mathcal{N} = 4$ U(N) SYMs theory. The particle content of this theory are: N^2 gauge fields, $6N^2$ massless scalars and $4N^2$ Weyl fermions, that is, we have $8N^2$ bosonic and $8N^2$ fermionic degrees of freedom. The free energy and hence the entropy can be calculated in the free field limit $g_{YM}^2 N \to 0$. The free energy of a gas of bosons and fermions in a box of volume $V^{(n)}$, the superscript denotes the number of spatial dimensions, calculated in the grand canonical ensemble is

$$F = -T \ln Z = T \sum_{s_i, p} s_i \ln \left(1 - s_i e^{-\beta p}\right) , \tag{18}$$

where s_i is +1 for bosons and -1 for fermions. Now taking the volume of the box to be very large implies $\sum_p = V^{(n)} \int d^n p/(2\pi)^n$. The free energy, then, becomes [13]

$$\begin{aligned} F &= T V^{(n)} \int \frac{d^n p}{(2\pi)^n} \sum_{s_i} \ln \left(1 - e^{(-\beta p + \ln s_i)}\right) \\ &= -\frac{T^{n+1} V^{(n)}}{2^n} \frac{1}{(\pi)^{n/2}} \frac{\Gamma(n+1)}{\Gamma(n/2+1)} (2 - 2^{-n}) \zeta(n+1) . \end{aligned} \tag{19}$$

For $\mathcal{N} = 4$, U(N) SYM theory in four dimension, the free energy is

$$F = -\frac{\pi^2 N^2 T^4 V^{(3)}}{6} . \tag{20}$$

Evaluating the specific heat from the free energy using the definitions as:

$$\begin{aligned} E &= F - T \left(\frac{dF}{dT}\right)_V , \\ C_{\rm CFT} &= \left(\frac{dE}{dT}\right)_V , \end{aligned} \tag{21}$$

and substituting in equation (11), we get:

$$\mathcal{S}_{\rm CFT} = S_{\rm CFT} - \frac{1}{2} \ln(S_{\rm CFT}) + \text{constant} , \tag{22}$$

where

$$S_{\rm CFT} = \frac{2\pi^2 N^2 T^3 V^{(3)}}{3} . \tag{23}$$

Here, $T = T_{\rm CFT}$ is clearly associated to the temperature of the gauge theory that resides on the boundary.

We would now like to compare the corrected entropy formula for the bulk (17) and for the boundary CFT (22) following AdS/CFT prescription. In order to do so, let us concentrate at an asymptotic distance $r \equiv L >> r_+$ of the AdS-Schwarzschild black hole where lives the gauge theory. In this region, the geometry of the above space-time becomes

$$ds^2 \sim L^2 \left[-\frac{dt^2}{l^2} + d\Omega_3^2\right] . \tag{24}$$

Due to red-shift, the temperature at this boundary therefore is

$$T_{\rm CFT} = \frac{T_{\rm BH}}{\sqrt{-g_{00}}} = \frac{l}{L} T_{\rm BH}\,, \tag{25}$$

where $T_{\rm BH}$ is the Hawking temperature associated with the black hole. Substituting (25) in (14) and using N^2 from the AdS/CFT dictionary [1],

$$N^2 = \frac{\pi l^3}{2G_5}\,, \tag{26}$$

we get the standard[3]

$$S_{\rm CFT} = \frac{2\pi^2 r_+^3}{3G_5} = \frac{4}{3} S_{\rm BH}\,. \tag{27}$$

This, in turn, implies that (22) can be written as

$$\mathcal{S}_{\rm CFT} = \frac{4}{3} S_{\rm BH} - \frac{1}{2} \ln\left(\frac{4}{3} S_{\rm BH}\right) + \text{constant}\,. \tag{28}$$

Let us now consider the case of rotating black hole in AdS space. The line element is [14],

$$\begin{aligned} ds^2 \;&= -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi} d\phi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{(r^2+a^2)}{\Xi} d\phi\right)^2 + \\ &\quad + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{\Delta_\theta}\, d\theta^2 + r^2\cos^2\theta\, d\psi^2\,, \end{aligned} \tag{29}$$

where $0 \le \phi, \psi \le 2\pi$ and $0 \le \theta \le \pi/2$, and

$$\begin{aligned} \Delta \;&= \left(r^2 + a^2\right)\left(1 + \frac{r^2}{l^2}\right) - 2MG_5\,, \\ \Delta_\theta \;&= 1 - \frac{a^2}{l^2}\cos^2\theta\,, \\ \rho^2 \;&= r^2 + a^2\cos^2\theta\,, \\ \Xi \;&= 1 - \frac{a^2}{l^2}\,. \end{aligned} \tag{30}$$

The horizon is at

$$r_+^2 = \frac{l^2}{2}\left(-\left(1 + \frac{a^2}{l^2}\right) + \sqrt{\left(1 + \frac{a^2}{l^2}\right)^2 + 4\frac{(2MG_5 - a^2)}{l^2}}\right)\,. \tag{31}$$

The entropy and temperature associated with the black hole is given by

$$S_{\rm BH} = \frac{\pi^2\left(r_+^2 + a^2\right) r_+}{2G_5\Xi}\,, \qquad T_{\rm BH} = \frac{r_+\left(1 + a^2/l^2 + 2r_+^2/l^2\right)}{2\pi\left(r_+^2 + a^2\right)}\,. \tag{32}$$

[3]Appearance of this extra 4/3 factor in the relation between entropy formulas is discussed in [13].

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The angular velocity at the horizon Ω and angular momentum J are given by

$$\Omega_H = \frac{a\Xi}{r_+^2 + a^2}\,, \qquad J = \frac{\pi M a}{2\Xi^2}\,. \tag{33}$$

In the region, $a \to 0$, the specific heat at constant angular momentum can easily be calculated and is given by

$$C = \frac{3\pi^2 r_+^5 \left\{9J^2\left(1+r_+^2\right) + 2\left(2r_+^2 + l^2\right)\tilde{M}^2\right\}}{4G_5 r_+^2\left(2r_+^2 - l^2\right)\tilde{M}^2 + 9G_5 J^2\left(r_+^2 + 3l^2\right)}\,, \tag{34}$$

where $\tilde{M}$ is the mass of the rotating black hole above the AdS background and is given by

$$\tilde{M} = \frac{3\pi}{4\Xi} M\,. \tag{35}$$

It follows clearly from (34), that for large enough $\tilde{M}$, there is a domain in which C is positive. Now, using (34) and (11), along with the bulk-boundary correspondence for rotating black hole in AdS space [15], it is easy to check that in high temperature limit, the relation (28) re-emerges.

To conclude, we have analysed the correction to entropy due to thermal fluctuations. We then applied our result to AdS black holes and compared it with what is expected from AdS/CFT correspondence. We found that the corrected entropy formula is of the form (1) with $c = 1/2$.

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